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RESEARCH ARTICLE

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Strong convergence theorems for generalized equilibrium, variational inequalities and nonlinear operators

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Abstract A new iterative scheme is introduced to approximate a common element of the solution set of a generalized mixed equilibrium problem, the solution set of a variational inequality problem, the set of common fixed points of two countable families of weak relatively nonexpansive mappings and the set of zeros of a maximal monotone operator in Banach spaces. The results obtained in this paper generalize and improve upon some existing results in recent literature.

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الملخص

يتم تقديم مخطط تكراري جديد لتقريب عنصر مشترك لمجموعة الحل لمسألة توازن مختلطة مُعَمَّمَة، ومجموعة الحل لمسألة متراجحة متغيرية، ومجموعة النقط الثابتة المشتركة لعائلتين قابلتين للعد من تطبيقات ضعيفة غير تمدُّدِيَّة نسبياً، ومجموعة أصفار مؤثر رتيب أعظمي في فضاءات باناخ. النتائج التي تم الحصول عليها في هذه الورقة تعمم وتحسن بعض النتائج الموجودة في الأبحاث الحديثة.

1 Introduction

Equilibrium problems influence the development of several branches of pure and applied sciences [2]. It has been shown that the theory of equilibrium problems provides a novel and unified treatment of a wide class of problems, which arise in economics, finance, physics, image reconstruction, ecology, transportation, network, elasticity and optimization.

Let E be a real Banach space with dual E^* and let C be a nonempty closed convex subset of E. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real valued function. A generalized mixed equilibrium

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problem (GMEP) [23] is formulated as finding a point $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0, \quad \forall\, y \in C \tag{1.1}$$

where $A : C \to E^*$ is some nonlinear operator. Here are some special cases of Problem (1.1).

If $A = 0$, then Problem (1.1) is reduced to the following mixed equilibrium problem [5] of finding $x \in C$ such that

$$\Theta(x, y) + \varphi(y) - \varphi(x) \geq 0, \quad \forall\, y \in C.$$

The set of solutions of this problem is denoted by MEP.

If $\varphi = 0$, then Problem (1.1) is reduced to the following generalized equilibrium problem [18] of finding $x \in C$ such that

$$\Theta(x, y) + \langle Ax, y - x\rangle \geq 0, \quad \forall\, y \in C.$$

If $\varphi = 0$ and $A = 0$, then Problem (1.1) is reduced to the following equilibrium problem of finding $x \in C$ such that

$$\Theta(x, y) \geq 0, \quad \forall\, y \in C. \tag{1.2}$$

The set of solutions of Problem (1.2) is denoted by EP.

If $\Theta = 0$ and $\varphi = 0$, then Problem (1.1) is reduced to the following classical variational inequality problem of finding $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall\, y \in C. \tag{1.3}$$

The set of solutions of Problem (1.3) is denoted by $\mathrm{VI}(C, A)$.

The problem (1.1) is general in the sense that it includes, as special cases, numerous problems in physics, optimization, variational inequalities, minimax problems, the Nash equilibrium problem in noncooperative games and others; see, for instance, [2,5,18–21].

The normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in E,$$

where $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing between E and E^*. It is well known that if E is smooth then J is single-valued, and if E is uniformly smooth then J is uniformly continuous on bounded subsets of E in the norm-to-norm topology. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual E^*, then J^{-1} is single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$ and $J^{-1}J = I_E$ (see [17]).

On the other hand, let $W : E \rightrightarrows E^*$ be a set-valued mapping. The problem of finding $v \in E$ satisfying $0 \in Wv$ contains numerous problems in economics, optimization and physics. Such $v \in E$ is called a zero of W.

A set-valued mapping $W : E \rightrightarrows E^*$ with graph $G(W) = \{(x, x^*) : x^* \in Wx\}$, domain $D(W) = \{x \in E : Wx \neq \emptyset\}$, and range $R(W) = \cup\{Wx : x \in D(W)\}$ is said to be monotone if $\langle x - y, x^* - y^*\rangle \geq 0$ for all $x^* \in Wx$, $y^* \in Wy$. A monotone operator W is said to be maximal monotone if the graph $G(W)$ of W is not properly contained in the graph of any other monotone operator. It is known that a monotone W is a maximal monotone if and only if $R(J + rW) = E^*$ for all $r > 0$ when E is a reflexive, strictly convex and smooth Banach space (see [13]).

Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E and $W : E \rightrightarrows E^*$ be a monotone operator satisfying $D(W) \subset C \subset J^{-1}(\cap_{r>0} R(J + rW))$. Then the resolvent of W defined by $J_r = (J + rW)^{-1}J$ is a single-valued mapping from E to $D(W)$ for all $r > 0$. For $r > 0$, the Yosida approximation of W is defined by $W_r x = (Jx - JJ_r x)/r$ for all $x \in E$.

Let $A : C \to E^*$ be a single-valued mapping. Recall that A is monotone if, for each $x, y \in C$,

$$\langle x - y, Ax - Ay\rangle \geq 0.$$

We say that A is γ-inverse strongly monotone (γ-ism) if there exists a positive real number $\gamma > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \gamma\|Ax - Ay\|^2, \quad \forall\, x, y \in C.$$

If A is γ-ism, then it is Lipschitz continuous with constant $1/\gamma$, i.e.,

$$\|Ax - Ay\| \le (1/\gamma)\|x - y\|, \quad x, y \in C.$$

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad x, y \in E,$$

is studied by Alber [1], Kamimura and Takahashi [7] and Reich [12]. It is not hard to find that

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2, \quad x, y \in E.$$

Observe that, in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$, $\forall\, x, y \in H$.

Lemma 1.1 [1] *Let C be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then there exists a unique element $x_0 \in C$ such that $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$.*

Let E be a reflexive, strictly convex, and smooth Banach space and C be a nonempty closed and convex subset of E. The generalized projection mapping, introduced by Alber [1], is a mapping $\Pi_C : E \to C$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x, y)$, that is $\Pi_C x = x_0$, where x_0 is determined as in Lemma 1.1, i.e., the unique solution in C to the minimization problem $\phi(x_0, x) = \min\{\phi(z, x) : z \in C\}$.

Let T be a mapping from C into itself. By $F(T)$ we denote the set of fixed points of T, i.e., $F(T) = \{x \in C : Tx = x\}$. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A mapping T from C into itself is called nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$, and relatively nonexpansive (see [3,4]) if $\widehat{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. A point p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called relatively weak nonexpansive if $\widetilde{F}(T) = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

When W is a maximal monotone, a well-known method for solving the equation $0 \in Wv$ in a Hilbert space H is the proximal point algorithm (see [14]): $x_1 = x \in H$ and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, \ldots, \tag{1.4}$$

where $\{r_n\} \subset (0, \infty)$ and $J_r = (I + rW)^{-1}$ for all $r > 0$ is the resolvent operator for W. Rockafellar [14] proved in the Hilbert space setting that the sequence $\{x_n\}$ converges weakly to an element of $W^{-1}0$.

The modifications of the proximal point algorithm for different operators have been investigated by many authors. Kohsaka and Takahashi [9] considered the following algorithm (1.5) in a smooth and uniformly convex Banach space:

$$x_{n+1} = J^{-1}(\beta_n J(x_1) + (1 - \beta_n) J(J_{r_n} x_n)), \quad n = 1, 2, \ldots, \tag{1.5}$$

and Kamimura, Kohsaka and Takahashi [8] considered the algorithm (1.6) in a uniformly smooth and uniformly convex Banach space:

$$x_{n+1} = J^{-1}(\beta_n J(x_n) + (1 - \beta_n) J(J_{r_n} x_n)), \quad n = 1, 2, \ldots. \tag{1.6}$$

They showed that the algorithm (1.5) converges strongly and the algorithm (1.6) converges weakly provided that the sequences $\{\beta_n\}$ and $\{r_n\}$ of real numbers are chosen appropriately.

Recently, Habtu and Naseer [22] introduced the following iterative scheme for finding a common element of the solution set of a variational inequality problem and a relatively weak nonexpansive mapping:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ y_n = \Pi_C J^{-1}(Jx_n - \alpha_n Ax_n), \\ z_n = Ty_n, \\ H_0 = \{v \in C : \phi(v, z_0) \le \phi(v, y_0) \le \phi(v, x_0)\}, \\ H_n = \{v \in H_{n-1} \cap W_{n-1} : \phi(v, z_n) \le \phi(v, y_n) \le \phi(v, x_n)\}, \\ W_0 = C, \\ W_n = \{v \in W_{n-1} \cap H_{n-1} : \langle x_n - v, Jx_0 - Jx_n\rangle \ge 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, n \ge 0. \end{cases}$$

In 2009, Takahashi and Zembayashi [19] proposed the following modification of iteration process for a relatively nonexpansive mapping:

$$\begin{cases} x_0 = x \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ u_n \in C \text{ such that } \Theta(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle \geq 0, \quad \forall\, y \in C, \\ C_n = \{z \in C : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x. \end{cases}$$

They proved that $\{x_n\}$ converges strongly to $\Pi_{F(T)\cap \mathrm{EP}} x$.

Very recently, Wang and Zeng [20] considered the following iterative scheme for approximating a common element of the solution set of a generalized mixed equilibrium problem, the solution set of a variational inequality problem and the set of fixed points of a relatively weak nonexpansive mapping:

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily}, \\ y_n = \Pi_C J^{-1}(Jx_n - \lambda_n A x_n), \\ z_n = J^{-1}(\beta_n J x_n + (1-\beta_n) J T y_n), \\ u_n = K_{r_n} z_n, \\ H_0 = \{v \in C : \phi(v, u_0) \leq \beta_0 \phi(v, x_0) + (1-\beta_0)\phi(v, y_0) \leq \phi(v, x_0)\}, \\ H_n = \{v \in H_{n-1} \cap W_{n-1} : \phi(v, u_n) \leq \beta_n \phi(v, x_n) + (1-\beta_n)\phi(v, y_n) \leq \phi(v, x_n)\}, \\ W_0 = C, \\ W_n = \{v \in W_{n-1} \cap H_{n-1} : \langle x_n - v, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{H_n \cap W_n} x_0, n \geq 0, \end{cases}$$

where $K_{r_n} : E \to C$ is a mapping (see its definition in the statement of Lemma 2.8 in Sect. 2). Under suitable conditions, they established some strong convergence theorems.

In 2010, Su et al. [16] introduced the definitions of countable family of relatively nonexpansive mappings and countable family of weak relatively nonexpansive mappings which are generalizations of relatively nonexpansive mappings and relatively weak nonexpansive mappings respectively.

Let C be a closed convex subset of E, and let $\{T_n\}_{n=0}^{\infty}$ be a countable family of mappings from C into itself. Let $F(T_n)$ denote the set of fixed points of T_n for all $n \geq 0$. A point p in C is said to be an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ [16] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. The set of asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ will be denoted $\widehat{F}(\{T_n\}_{n=0}^{\infty})$. A point p in C is said to be a strong asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$ [16] if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$. The set of strong asymptotic fixed points of $\{T_n\}_{n=0}^{\infty}$ will be denoted $\widetilde{F}(\{T_n\}_{n=0}^{\infty})$.

Definition 1.2 [16] A countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be relatively nonexpansive if the following conditions are satisfied:

(1) $\cap_{n=0}^{\infty} F(T_n) \neq \emptyset$;

(2) $\phi(u, T_n x) \leq \phi(u, x), \quad \forall u \in F(T_n),\ x \in C,\ n \geq 0$;

(3) $\widehat{F}(\{T_n\}_{n=0}^{\infty}) = \cap_{n=0}^{\infty} F(T_n)$.

Definition 1.3 [16] Countable family of mappings $\{T_n\}_{n=0}^{\infty}$ is said to be weak relatively nonexpansive if the following conditions are satisfied:

(1) $\cap_{n=0}^{\infty} F(T_n) \neq \emptyset$;
(2) $\phi(u, T_n x) \leq \phi(u, x), \quad \forall u \in F(T_n),\ x \in C,\ n \geq 0$;
(3) $\widetilde{F}(\{T_n\}_{n=0}^{\infty}) = \cap_{n=0}^{\infty} F(T_n)$.

It is obvious that a countable family of relatively nonexpansive mappings is a countable family of weak relatively nonexpansive mappings. In fact, for any countable family of mappings $\{T_n\}_{n=0}^{\infty}$, we have $\cap_{n=0}^{\infty} F(T_n) \subset \widetilde{F}(\{T_n\}_{n=0}^{\infty}) \subset \widehat{F}(\{T_n\}_{n=0}^{\infty})$. Therefore, if $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively nonexpansive mappings, then $\cap_{n=0}^{\infty} F(T_n) = \widetilde{F}(\{T_n\}_{n=0}^{\infty}) = \widehat{F}(\{T_n\}_{n=0}^{\infty})$. But there is a countable family of weak relatively nonexpansive mappings which is not a countable family of relatively nonexpansive mappings (see [16]). Moreover they established the following convergence theorems:

Theorem SXZ1 [16] *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a non-empty closed convex subset of E, let $\{T_n\}, \{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $F := (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C, \ \textit{chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\}, \\ C_0 = \{z \in C : \phi(v, y_0) \le \phi(v, x_0)\}, \\ Q_n = \{z \in Q_{n-1} \cap Q_{n-1} : \langle x_n - v, J x_0 - J x_n \rangle \ge 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, n \ge 0, \end{cases}$$

with the conditions:

(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
(iii) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0, 1)$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Theorem SXZ2 [16] *Let E be a uniformly convex and uniformly smooth real Banach space, let C be a non-empty closed convex subset of E, let $\{T_n\}, \{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $F := (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by the following algorithm:*

$$\begin{cases} x_0 \in C, \ \textit{chosen arbitrarily}, \\ z_n = J^{-1}(\beta_n^{(1)} J x_0 + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n), \\ y_n = J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \\ \qquad \phi(z, y_n) \le (1 - \alpha_n \beta_n^{(1)}) \phi(z, x_n) + \alpha_n \beta_n^{(1)} \phi(z, x_0)\}, \\ C_0 = \{z \in C : \phi(v, y_0) \le \phi(v, x_0)\}, \\ Q_n = \{z \in Q_{n-1} \cap Q_{n-1} : \langle x_n - v, J x_0 - J x_n \rangle \ge 0\}, \\ Q_0 = C, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, n \ge 0, \end{cases}$$

with the conditions:

(i) $\lim_{n\to\infty} \beta_n^{(1)} = 0$;
(ii) $\limsup_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection from C onto F.

Motivated and inspired by the above work, we are aimed in this paper at introducing a new hybrid projection iterative scheme which will be proved to converge strongly to a common element of the solution set of a generalized mixed equilibrium problem, the solution set of a variational inequality problem and the set of common fixed points of two countable families of weak relatively nonexpansive mappings in Banach spaces. Our results improve and generalize many existing results in current literature.

2 Preliminaries

Let E be a real normed linear space. The modulus of smoothness of E is the function $\rho_E : [0, +\infty) \to [0, +\infty)$ defined by

$$\rho_E(\tau) := \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$

The space E is said to be smooth if $\rho_E(\tau) > 0,\ \forall\ \tau > 0$, and E is called uniformly smooth if and only if $\lim_{t\to 0^+} \frac{\rho_E(t)}{t} = 0$.

The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\varepsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1, \varepsilon = \|x - y\|\right\}.$$

E is called uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0, 2]$. Let $p > 1$; then E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) \geq c\varepsilon^p$ for every $\varepsilon \in (0, 2]$. Observe that every p-uniformly convex is uniformly convex. It is well known (see for example [21]) that

$$L_p(l_p) \text{ or } W_m^p \text{ is } \begin{cases} p\text{-uniformly convex if } p \geq 2, \\ 2\text{- uniformly convex if } 1 < p \leq 2. \end{cases}$$

In what follows, we shall make use of the following lemmas.

Lemma 2.1 [21] *Let E be a 2-uniformly convex and smooth Banach space. Then, for all $x, y \in E$, we have*

$$\|x - y\| \leq \frac{2}{c^2}\|Jx - Jy\|, \tag{2.1}$$

where J is the normalized duality mapping of E and $1/c$ $(0 < c \leq 1)$ is the 2-uniform convexity constant of E.

Lemma 2.2 [1,7] *Let E be a real smooth, strictly convex, and reflexive Banach space and C be a nonempty closed convex subset. Then the following conclusions hold:*

(1) $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x)$ *for all* $x \in E,\ y \in C$,
(2) *Suppose $x \in E$ and $z \in C$, then*

$$z = \Pi_C x \quad \Leftrightarrow \quad \langle z - y, Jx - Jz\rangle \geq 0, \quad \forall\, y \in C.$$

Lemma 2.3 [7] *Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$ as $n \to \infty$.*

Lemma 2.4 [11] *Let E be a real smooth Banach space and let $A : E \rightrightarrows E^*$ be a maximal monotone mapping. Then $A^{-1}(0)$ is a closed and convex subset of E.*

We denote by $N_C(v)$ the normal cone to C at a point $v \in C$, that is

$$N_C(v) := \{x^* \in E^* : \langle v - y, x^*\rangle \geq 0, \quad y \in C\}.$$

In the following, we shall use the following Lemma.

Lemma 2.5 [14] *Let C be a nonempty closed convex subset of a Banach space E and let A be a monotone and hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be the operator defined by*

$$Tv := \begin{cases} Av + N_C(v), & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

We will make use of the function $V : E \times E^* \to \mathbb{R}$ defined by [1]

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x\|^2, \quad x \in E,\ x^* \in E^*.$$

That is, $V(x, x^*) = \phi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. We know the following Lemma.

Lemma 2.6 [1] *Let E be a reflexive strictly convex and smooth Banach space. Then*

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^*\rangle \leq V(x, x^* + y^*), \quad x \in E,\ x^*, y^* \in E^*.$$

The following inequality is an extension to an inequality established in [21].

Lemma 2.7 [6] *Let E be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \le r\}$ be a closed ball of E. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z \in B_r(0)$ and $\lambda, \mu, \gamma \in [0, 1]$ with $\lambda + \mu + \gamma = 1$.

For solving the equilibrium problem, let us assume that Θ satisfies the following conditions:

(A1) $\Theta(x, x) = 0$ for all $x \in C$;
(A2) Θ is monotone, i.e., $\Theta(x, y) + \Theta(y, x) \le 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\lim_{t \to 0} \Theta(tz + (1 - t)x, y) \le \Theta(x, y);$$

(A4) for each $x \in C$, $y \mapsto \Theta(x, y)$ is convex and lower semi-continuous.

Lemma 2.8 [23] *Let C be a closed subset of a smooth, strictly convex and reflexive Banach space E. Let $B : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function, and $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then, for $r > 0$ and $x \in E$, there exists $u \in C$ such that*

$$\Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \ge 0, \quad \forall\, y \in C.$$

Define a mapping $K_r : E \to C$ by

$$K_r(x) := \{u \in C : \Theta(u, y) + \langle Bu, y - u\rangle + \varphi(y) - \varphi(u) + \frac{1}{r}\langle y - u, Ju - Jx\rangle \ge 0,\ \forall\, y \in C\}$$

for all $x \in E$. Then, the following conclusions hold:

(1) *K_r is single-valued;*
(2) *K_r is firmly nonexpansive, i.e., for all $x, y \in E$,*

$$\langle K_r x - K_r y, JK_r x - JK_r y\rangle \le \langle K_r x - K_r y, Jx - Jy\rangle;$$

(3) *$F(K_r) = GMEP$;*
(4) *GMEP is closed and convex;*
(5) *$\phi(p, K_r z) + \phi(K_r z, z) \le \phi(p, z),\ \forall\, p \in F(K_r),\ z \in E$.*

Lemma 2.9 [10] *Let E be a smooth, strictly convex and reflexive Banach space, let C be a nonempty closed convex subset of E, and let $W : E \rightrightarrows E^*$ be a monotone operator satisfying $D(W) \subset C \subset J^{-1}(\cap_{r>0} R(J + rW))$. Let $r > 0$, let J_r and W_r be the resolvent and the Yosida approximation of W, respectively. Then the following hold:*

(i) *$\phi(u, J_r x) + \phi(J_r x, x) \le \phi(u, x)$, for all $x \in C$, $u \in W^{-1}0$;*
(ii) *$(J_r x, W_r x) \in G(W)$, for all $x \in C$;*
(iii) *$F(J_r) = W^{-1}0$.*

3 Strong convergence theorems

Theorem 3.1 *Let E be a real uniformly smooth and 2-uniformly convex Banach space (e.g., L^p for $1 < p \le 2$) and C be a nonempty, closed and convex subset of E. Let $W : E \rightrightarrows E^*$ be a maximal monotone operator satisfying $D(W) \subset C$ and let $J_t = (J + tW)^{-1}J$ for all $t > 0$. Let $A : C \to E^*$ be a γ-inverse strongly monotone mapping, $B : C \to E^*$ be a monotone continuous mapping. Let $\{T_n\}$, $\{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n)) \cap VI(C, A) \cap GMEP \cap W^{-1}0 \ne \emptyset$. Assume that $\|Ax\| \le \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C, A)$.*

Suppose that $0 < a < \lambda_n < b := c^2\gamma/2$, *where* c *is the constant in* (2.1). *Let* $\{r_n\} \subset [c^*, +\infty)$ *for some* $c^* > 0$ *and* $\{t_n\} \subset (0, +\infty)$ *satisfy* $\liminf_{n\to\infty} t_n > 0$. *Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_0 \in C, \text{chosen arbitrarily}, \\ v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n J_{t_n} v_n + \beta_n^{(3)} JS_n v_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) Jz_n) \\ u_n = K_{r_n} y_n, \\ C_0 = \{z \in C : \phi(z, u_0) \le \phi(z, x_0)\}, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \le \phi(z, x_n)\}, \\ Q_0 = C, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, n \ge 0, \end{cases} \tag{$\diamond$}$$

where J *is the normalized duality mapping,* $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$, $\{\beta_n^{(3)}\}$ *and* $\{\alpha_n\}$ *are four sequences in [0, 1] satisfying* $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ *for all* $n \ge 0$, *and, in addition, the following conditions:*

(i) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(2)} > 0$;
(ii) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(3)} > 0$;
(iii) $\limsup_{n\to\infty} \alpha_n < 1$ *and* $\limsup_{n\to\infty} \beta_n^{(1)} < 1$.

Then $\{x_n\}$ *converges strongly to* $\Pi_\Omega x_0$.

Proof We divide the proof into four steps.

Step 1 First we prove that $\Omega \subset C_n \cap Q_n,\ \forall\, n \ge 0$.

In fact, it is obvious that Ω is closed and convex and it follows from the definitions of C_n and Q_n that they both are closed and convex for each $n \ge 0$. Next, we prove by induction that $\Omega \subset C_n \cap Q_n$ for all $n \ge 0$. Observe that $\Omega \subset Q_0 = C$. Let $w_n = J_{t_n} v_n$, for any given $p \in \Omega$, from the definition of $\phi(x, y)$ and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \phi(p, y_0) &= \phi(p, J^{-1}(\alpha_0 Jx_0 + (1-\alpha_0)Jz_0)) \\ &= \|p\|^2 - 2\langle p, \alpha_0 Jx_0 + (1-\alpha_0)Jz_0\rangle + \|\alpha_0 Jx_0 + (1-\alpha_0)Jz_0\|^2 \\ &\le \alpha_0\phi(p, x_0) + (1-\alpha_0)\phi(p, z_0), \end{aligned} \tag{3.1}$$

$$\begin{aligned} \phi(p, z_0) &= \phi(p, J^{-1}(\beta_0^{(1)} Jx_0 + \beta_0^{(2)} JT_0 w_0 + \beta_0^{(3)} JS_0 v_0)) \\ &= \|p\|^2 - 2\langle p, \beta_0^{(1)} Jx_0 + \beta_0^{(2)} JT_0 w_0 + \beta_0^{(3)} JS_0 v_0)\rangle \\ &\quad + \|\beta_0^{(1)} Jx_0 + \beta_0^{(2)} JT_0 w_0 + \beta_0^{(3)} JS_0 v_0)\|^2 \\ &\le \beta_0^{(1)}\phi(p, x_0) + \beta_0^{(2)}\phi(p, T_0 w_0) + \beta_0^{(3)}\phi(p, S_0 v_0) \\ &\le \beta_0^{(1)}\phi(p, x_0) + \beta_0^{(2)}\phi(p, w_0) + \beta_0^{(3)}\phi(p, v_0). \end{aligned} \tag{3.2}$$

Moreover it follows from Lemmas 2.2 and 2.6 that

$$\begin{aligned} \phi(p, v_0) &\le \phi(p, J^{-1}(Jx_0 - \lambda_0 Ax_0)) \\ &\le V(p, Jx_0 - \lambda_0 Ax_0) \\ &\le V(p, Jx_0 - \lambda_0 Ax_0 + \lambda_0 Ax_0) - 2\langle J^{-1}(Jx_0 - \lambda_0 Ax_0) - p, \lambda_0 Ax_0\rangle \\ &= \phi(p, x_0) - 2\lambda_0\langle x_0 - p, Ax_0 - Ap\rangle - 2\lambda_0\langle x_0 - p, Ap\rangle \\ &\quad -2\lambda_0\langle J^{-1}(Jx_0 - \lambda_0 Ax_0) - x_0, Ax_0\rangle. \end{aligned}$$

Since $p \in \mathrm{VI}(C, A)$, A is γ-ism, from the above inequality, Lemma 2.1 and the fact that $\|Ax\| \le \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C, A)$, we obtain

$$\begin{aligned}\phi(p, v_0) &\le \phi(p, x_0) - 2\lambda_0\gamma\|Ax_0 - Ap\|^2 + 2\lambda_0\|J^{-1}(Jx_0 - \lambda_0 Ax_0) - x_0\|\|Ax_0\| \\ &\le \phi(p, x_0) - 2\lambda_0\gamma\|Ax_0 - Ap\|^2 + \frac{4}{c^2}\lambda_0^2\|Ax_0 - Ap\|^2 \\ &= \phi(p, x_0) + 2\lambda_0\left(\frac{2}{c^2}\lambda_0 - \gamma\right)\|Ax_0 - Ap\|^2 \\ &\le \phi(p, x_0). \end{aligned} \tag{3.3}$$

It follows from Lemma 2.9 that

$$\phi(p, w_0) = \phi(p, J_{t_0}v_0) \le \phi(p, v_0). \tag{3.4}$$

From (3.2)–(3.4), we have

$$\phi(p, z_0) \le \phi(p, x_0). \tag{3.5}$$

So from (3.1) and (3.5), we have

$$\phi(p, y_0) \le \phi(p, x_0). \tag{3.6}$$

By Lemma 2.8(5) and (3.6), we have

$$\phi(p, u_0) = \phi(p, K_{r_0}y_0) \le \phi(p, y_0) \le \phi(p, x_0). \tag{3.7}$$

Therefore, $p \in C_0$ and so, $p \in C_0 \cap Q_0$. Suppose $\Omega \subset C_{n-1} \cap Q_{n-1}$. Then we deduce from (3.1)–(3.7) that

$$\phi(p, w_n) \le \phi(p, v_n) \le \phi(p, x_n), \tag{3.8}$$

$$\begin{aligned}\phi(p, z_n) &\le \beta_n^{(1)}\phi(p, x_n) + \beta_n^{(2)}\phi(p, v_n) + \beta_n^{(3)}\phi(p, v_n) \\ &\le \beta_n^{(1)}\phi(p, x_n) + (1 - \beta_n^{(1)})[\phi(p, x_n) - 2\lambda_n\gamma\|Ax_n - Ap\|^2 + \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2] \\ &= \phi(p, x_n) + (1 - \beta_n^{(1)})2\lambda_n(\frac{2}{c^2}\lambda_n - \gamma)\|Ax_n - Ap\|^2 \\ &\le \phi(p, x_n), \end{aligned} \tag{3.9}$$

$$\begin{aligned}\phi(p, z_n) &\le \beta_n^{(1)}\phi(p, x_n) + \beta_n^{(2)}\phi(p, w_n) + \beta_n^{(3)}\phi(p, v_n) \\ &\le (1 - \beta_n^{(2)})\phi(p, x_n) + \beta_n^{(2)}\phi(p, w_n), \end{aligned} \tag{3.10}$$

$$\phi(p, u_n) \le \phi(p, y_n) \le \alpha_n\phi(p, x_n) + (1 - \alpha_n)\phi(p, z_n) \le \phi(p, x_n). \tag{3.11}$$

By (3.11) we see that $p \in C_n$. From Lemma 2.2 and $x_n = \Pi_{C_{n-1}\cap Q_{n-1}}x_0$, we have

$$\langle p - x_n, Jx_0 - Jx_n\rangle \le 0,$$

which implies that $p \in Q_n$. Hence $\Omega \subset C_n \cap Q_n$. By induction, $\Omega \subset C_n \cap Q_n$ for each $n \ge 0$. So the sequence $\{x_n\}$ generated by ($\diamondsuit$) is well defined for each $n \ge 0$.

Step 2 We prove that $\|w_n - T_nw_n\| \to 0$, $\|v_n - S_nv_n\| \to 0 (n \to \infty)$.

Repeating the argument of Step 2 in the proof of [9, Theorem 3.1], we have $\phi(x_n, x_m) \to 0$ $(n, m \to \infty)$. Thus $\{x_n\}$ is a Cauchy sequence and let $x^* \in C$ be its limit, that is, $x_n \to x^*$ $(n \to \infty)$; in particular, $\{x_n\}$ is bounded. Using Lemma 2.3, we have

$$\|x_n - x_{n+1}\| \to 0 \quad (n \to \infty). \tag{3.12}$$

Observe that by (3.8)–(3.11) the boundedness of $\{x_n\}$ implies that the sequences $\{u_n\}$, $\{z_n\}$, $\{v_n\}$, $\{w_n\}$, $\{y_n\}$, $\{T_nw_n\}$, and $\{S_nv_n\}$ are all bounded. Since $x_{n+1} = \Pi_{C_n\cap Q_n}x_0 \in C_n$, we have

$$\phi(x_{n+1}, u_n) \le \phi(x_{n+1}, x_n) \to 0 \quad (n \to \infty),$$

which together with Lemma 2.3 imply that $\|x_{n+1} - u_n\| \to 0$ $(n \to \infty)$. So

$$\|x_n - u_n\| \to 0 \quad (n \to \infty). \tag{3.13}$$

By Lemma 2.8(5), (3.11), (3.13) and the uniform continuity of J over bounded sets, we have, for any $p \in \Omega$,

$$\begin{aligned}\phi(u_n, y_n) &= \phi(K_{r_n} y_n, y_n)\\ &\le \phi(p, y_n) - \phi(p, K_{r_n} y_n)\\ &\le \phi(p, x_n) - \phi(p, u_n)\\ &= \|x_n\|^2 - \|u_n\|^2 - 2\langle p, Jx_n - Ju_n\rangle\\ &\le (\|x_n - u_n\|)(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\| \to 0.\end{aligned}$$

Furthermore from Lemma 2.3 we have

$$\|y_n - u_n\| \to 0 \quad (n \to \infty). \tag{3.14}$$

So by (3.13) and (3.14) we have

$$\|y_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.15}$$

From (◇), (3.15) and and again the uniform continuity of J on bounded sets, we get

$$\|Jy_n - Jx_n\| = (1 - \alpha_n)\|Jz_n - Jx_n\| \to 0.$$

Combining the above inequality and the condition (iii), we have

$$\|Jz_n - Jx_n\| \to 0 \quad (n \to \infty).$$

Moreover since J^{-1} is uniformly continuous on bounded sets, we have

$$\|z_n - x_n\| \to 0 \quad (n \to \infty). \tag{3.16}$$

It follows from Lemma 2.7, (3.8) and the boundedness of $\{x_n\}$ and $\{T_n w_n\}$ that, for any $p \in \Omega$,

$$\begin{aligned}\phi(p, z_n) &= \phi(p, J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n w_n + \beta_n^{(3)} JS_n v_n))\\ &= \|p\|^2 - 2\langle p, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n w_n + \beta_n^{(3)} JS_n v_n\rangle\\ &\quad + \|\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n w_n + \beta_n^{(3)} JS_n v_n\|^2\\ &\le \beta_n^{(1)}\phi(p, x_n) + \beta_n^{(2)}\phi(p, T_n w_n) + \beta_n^{(3)}\phi(p, S_n v_n) - \beta_n^{(1)}\beta_n^{(2)} g(\|Jx_n - JT_n w_n\|)\\ &\le \beta_n^{(1)}\phi(p, x_n) + \beta_n^{(2)}\phi(p, w_n) + \beta_n^{(3)}\phi(p, v_n) - \beta_n^{(1)}\beta_n^{(2)} g(\|Jx_n - JT_n w_n\|)\\ &\le \phi(p, x_n) - \beta_n^{(1)}\beta_n^{(2)} g(\|Jx_n - JT_n w_n\|).\end{aligned} \tag{3.17}$$

Hence from (3.16) and (3.17), we have

$$\beta_n^{(1)}\beta_n^{(2)} g(\|Jx_n - JT_n w_n\|) \le \phi(p, x_n) - \phi(p, z_n) \to 0 \quad (n \to \infty).$$

Using the same argument, we obtain

$$\beta_n^{(1)}\beta_n^{(3)} g(\|Jx_n - JS_n v_n\|) \le \phi(p, x_n) - \phi(p, z_n) \to 0 \quad (n \to \infty).$$

Moreover from the conditions (i), (ii) and the properties of g, we have

$$\|Jx_n - JT_n w_n\| \to 0 \quad (n \to \infty),$$

and

$$\|Jx_n - JS_n v_n\| \to 0 \quad (n \to \infty).$$

Since J^{-1} is uniformly continuous on bounded sets, we obtain

$$\|x_n - T_n w_n\| \to 0 \quad (n \to \infty), \tag{3.18}$$

$$\|x_n - S_n v_n\| \to 0 \quad (n \to \infty). \tag{3.19}$$

By (3.9) and (3.16) we have

$$(1-\beta_n^{(1)})2\lambda_n\left(\gamma-\frac{2}{c^2}\lambda_n\right)\|Ax_n-Ap\|^2\le\phi(p,x_n)-\phi(p,z_n)\to 0\quad(n\to\infty).$$

So from the condition (iii), we have

$$\|Ax_n-Ap\|\to 0\quad(n\to\infty). \tag{3.20}$$

From Lemmas 2.1, 2.2, 2.6, (3.20) and the fact that $\|Ax\|\le\|Ax-Ap\|$ for all $x\in C$ and $p\in VI(C,A)$, we have

$$\begin{aligned}\phi(x_n,v_n)&=\phi(x_n,\Pi_C J^{-1}(Jx_n-\lambda_nAx_n))\\&\le\phi(x_n,J^{-1}(Jx_n-\lambda_nAx_n))\\&=V(x_n,Jx_n-\lambda_nAx_n)\\&\le V(x_n,Jx_n-\lambda_nAx_n+\lambda_nAx_n)-2\langle J^{-1}(Jx_n-\lambda_nAx_n)-x_n,\lambda_nAx_n\rangle\\&=\phi(x_n,x_n)+2\lambda_n\|J^{-1}(Jx_n-\lambda_nAx_n)-J^{-1}Jx_n\|\|Ax_n\|\\&\le 2\lambda_n^2\frac{2}{c^2}\|Ax_n\|^2\\&\le 2\lambda_n^2\frac{2}{c^2}\|Ax_n-Ap\|^2\to 0\quad(n\to\infty).\end{aligned}$$

This implies that

$$\|x_n-v_n\|\to 0\quad(n\to\infty). \tag{3.21}$$

From (3.10), for any $p\in\Omega$, we have $\phi(p,w_n)\ge(1/\beta_n^{(2)})(\phi(p,z_n)-(1-\beta_n^{(2)})\phi(p,x_n))$, so by Lemma 2.9 and (3.8), we obtain

$$\begin{aligned}\phi(w_n,v_n)&=\phi(J_{t_n}v_n,v_n)\\&\le\phi(p,v_n)-\phi(p,J_{t_n}v_n)\\&\le\phi(p,x_n)-\phi(p,w_n)\\&\le\phi(p,x_n)-((1/\beta_n^{(2)})(\phi(p,z_n)-(1-\beta_n^{(2)})\phi(p,x_n)))\\&=\frac{1}{\beta_n^{(2)}}(\phi(p,x_n)-\phi(p,z_n))\\&\le\frac{1}{\beta_n^{(2)}}(\|x_n\|^2-\|z_n\|^2-2\langle p,Jx_n-Jz_n\rangle)\\&\le\frac{1}{\beta_n^{(2)}}(\|x_n-z_n\|(\|x_n\|+\|z_n\|)+2\|p\|\|Jx_n-Jz_n\|).\end{aligned} \tag{3.22}$$

Since the condition (i) implies that $\liminf_{n\to\infty}\beta_n^{(2)}>0$, it follows from (3.16) and (3.22) that

$$\lim_{n\to\infty}\phi(w_n,v_n)=0,$$

which implies that

$$\lim_{n\to\infty}\|w_n-v_n\|=0. \tag{3.23}$$

Combining (3.21) and (3.23), we get

$$\lim_{n\to\infty}\|x_n-w_n\|=0. \tag{3.24}$$

It follows from (3.18), (3.19), (3.21) and (3.24) that

$$\begin{aligned}\|w_n-T_nw_n\|&\to 0\quad(n\to\infty),\\\|v_n-S_nv_n\|&\to 0\quad(n\to\infty).\end{aligned} \tag{3.25}$$

Since $x_n \to x^*$ $(n \to \infty)$, from (3.21) and (3.24) we have $v_n \to x^*$, $w_n \to x^*$ $(n \to \infty)$. So by (3.25), we have $x^* \in (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n))$.

Step 3 We show that $x^* \in \mathrm{VI}(C, A) \cap \mathrm{GMEP} \cap W^{-1}0$.

Define an operator $S \subset E \times E^*$ by

$$Sv := \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

By Lemma 2.5, S is maximal monotone and $S^{-1}(0) = VI(C, A)$. Let $(v, w) \in G(S)$. Since $w \in Sv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. Moreover, $v_n \in C$ implies that

$$\langle v - v_n, w - Av\rangle \geq 0. \tag{3.26}$$

On the other hand, it follows from the fact that $v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n)$ and Lemma 2.2 that

$$\langle v - v_n, Jv_n - (Jx_n - \lambda_n Ax_n)\rangle \geq 0,$$

and hence

$$\left\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} - Ax_n \right\rangle \leq 0. \tag{3.27}$$

So from (3.26) and (3.27) we obtain

$$\begin{aligned}
\langle v - v_n, w\rangle &\geq \langle v - v_n, Av\rangle \\
&\geq \langle v - v_n, Av\rangle + \left\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} - Ax_n \right\rangle \\
&= \left\langle v - v_n, Av - Ax_n + \frac{Jx_n - Jv_n}{\lambda_n} \right\rangle \\
&\geq \langle v - v_n, Av - Av_n\rangle + \langle v - v_n, Av_n - Ax_n\rangle + \left\langle v - v_n, \frac{Jx_n - Jv_n}{\lambda_n} \right\rangle \\
&\geq -\|v - v_n\|\|Av_n - Ax_n\| - \|v - v_n\|\|\frac{Jx_n - Jv_n}{a}\|.
\end{aligned} \tag{3.28}$$

Since J is uniformly continuous on bounded sets and A is continuous, by (3.21) and (3.28) we have $\langle v - x^*, w\rangle \geq 0$ $(n \to \infty)$. Thus $x^* \in S^{-1}(0)$ and hence $x^* \in \mathrm{VI}(C, A)$.

Next we show that $x^* \in \mathrm{GMEP} = F(K_r)$. To see this, let

$$H(u_n, y) = \Theta(u_n, y) + \langle Bu_n, y - u_n\rangle + \varphi(y) - \varphi(u_n), \quad y \in C.$$

From (3.14) and (3.15), we have $u_n \to x^*$, $y_n \to x^*$ $(n \to \infty)$. Since J is uniformly continuous on bounded sets, from (3.14) we have $\lim_{n\to\infty} \|Ju_n - Jy_n\| = 0$. Therefore it follows from the assumption $r_n \geq c^*$ that $\lim_{n\to\infty} \frac{\|Ju_n - Jy_n\|}{r_n} = 0$. Since $u_n = K_{r_n} y_n$, we have

$$H(u_n, y) + \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall\, y \in C.$$

Combining the above inequality and (A2), we get

$$\|y - u_n\| \frac{\|Ju_n - Jy_n\|}{r_n} \geq \frac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq -H(u_n, y) \geq H(y, u_n), \quad \forall\, y \in C.$$

Taking the limit as $n \to \infty$ in the above inequality and by (A4) we have $H(y, x^*) \leq 0$, $\forall\, y \in C$. For any $t \in (0, 1)$ and $y \in C$, define $y_t = ty + (1 - t)x^* \in C$. So $H(y_t, x^*) \leq 0$. From (A1) and (A4), we have

$$0 = H(y_t, y_t) \leq tH(y_t, y) + (1 - t)H(y_t, x^*) \leq tH(y_t, y),$$

i.e., $H(y_t, y) \geq 0$. Thus by (A3) and letting $t \to 0$, we get $H(x^*, y) \geq 0$, $\forall\, y \in C$, which implies that $x^* \in$ GMEP.

Finally we prove that $x^* \in W^{-1}0$. Since J is uniformly continuous on bounded sets, from (3.23), we have

$$\lim_{n\to\infty} \|Jw_n - Jv_n\| = 0.$$

Since $\liminf_{n\to\infty} t_n > 0$, we get

$$\lim_{n\to\infty} \frac{1}{t_n}\|Jw_n - Jv_n\| = 0.$$

Hence

$$\|W_{t_n} v_n\| = \frac{1}{t_n}\|Jv_n - JJ_{t_n} v_n\| = \frac{1}{t_n}\|Jv_n - Jw_n\| \to 0 \quad (n \to \infty).$$

From Lemma 2.9(ii), we have $(w_n, W_{t_n} v_n) \in G(W)$. So by the monotonicity of W, for any $(z, z^*) \in G(W)$, we have $\langle z - w_n, z^* - W_{t_n} v_n\rangle \geq 0$ for all $n \geq 0$. Letting $n \to \infty$, we have $\langle z - x^*, z^*\rangle \geq 0$. Since W is maximal monotone, $x^* \in W^{-1}0$. Hence by Steps 2 and 3, we have $x^* \in \Omega$.

Step 4 Finally we prove that $x^* = \Pi_\Omega x_0$.

Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0$, we have

$$\langle x_{n+1} - z, Jx_0 - Jx_{n+1}\rangle \geq 0, \quad \forall\, z \in C_n \cap Q_n. \tag{3.29}$$

Taking the limit in (3.29) and as $\Omega \subset C_n \cap Q_n$ for each $n \geq 0$, we obtain

$$\langle x^* - z, Jx_0 - Jx^*\rangle \geq 0, \quad \forall\, z \in \Omega.$$

Therefore it follows from Lemma 2.2 that $x^* = \Pi_\Omega x_0$. □

If $A \equiv 0$, then we have the following result from Theorem 3.1.

Corollary 3.2 *Let E be a real uniformly smooth and uniformly convex Banach space and C be a nonempty, closed and convex subset of E. Let $W : E \rightrightarrows E^*$ be a maximal monotone operator satisfying $D(W) \subset C$ and let $J_t = (J + tW)^{-1}J$ for all $t > 0$. Let $B : C \to E^*$ be a monotone continuous mapping. Let $\{T_n\}, \{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n)) \cap GMEP \cap W^{-1}0 \neq \emptyset$. Let $\{r_n\} \subset [c^*, +\infty)$ for some $c^* > 0$ and $\{t_n\} \subset (0, +\infty)$ satisfy $\liminf_{n\to\infty} t_n > 0$. Let $\{x_n\}$ be the sequence generated by the algorithm*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily,} \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n J_{t_n} x_n + \beta_n^{(3)} JS_n x_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n) \\ u_n = K_{r_n} y_n, \\ C_0 = \{z \in C : \phi(z, u_0) \leq \phi(z, x_0)\}, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_0 = C, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \ n \geq 0, \end{cases}$$

where $\{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \{\beta_n^{(3)}\}$ and $\{\alpha_n\}$ are sequences in [0,1] satisfying the conditions:

(i) $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ *for all* $n \geq 0$,
(ii) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(2)} > 0$,
(iii) $\liminf_{n\to\infty} \beta_n^{(1)}\beta_n^{(3)} > 0$,
(iv) $\limsup_{n\to\infty} \alpha_n < 1$.

Then $\{x_n\}$ *converges strongly to* $\Pi_\Omega x_0$.

Proof From the proof of Theorem 3.1, (2.1) is used to prove (3.3), (3.9) and (3.21). Since $A \equiv 0$, $v_n = x_n$ for all $n \geq 0$. So in the uniformly smooth and uniformly convex Banach space X, we have $\phi(p, z_n) \leq \phi(p, x_n)$. Moreover, (3.18) and (3.19) are reduced to

$$\|x_n - T_n w_n\| \to 0 \quad \text{and} \quad \|x_n - S_n x_n\| \to 0 \qquad (n \to \infty).$$

So the condition $\limsup_{n\to\infty} \beta_n^{(1)} < 1$ in the assumption (iii) of Theorem 3.1 is superfluous in this case, and the remainder of the proof follows that of Theorem 3.1. □

Remark 3.3 Corollary 3.2 improves and generalizes Theorem 3.15 in [16].

If $A \equiv 0$, $T_n = T$, $S_n = S$, $W \equiv 0$, then we have the following result from Corollary 3.2.

Corollary 3.4 *Let E be a real uniformly smooth and uniformly convex Banach space and C be a nonempty, closed and convex subset of E. Let $B : C \to E^*$ be a monotone continuous mapping. Let T, S be two relatively weak nonexpansive mappings from C into itself such that $\Omega := F(T) \cap F(S) \cap GMEP \neq \emptyset$. Let $\{r_n\} \subset [c^*, +\infty)$ for some $c^* > 0$. Let $\{x_n\}$ be the sequence generated by the algorithm*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily,} \\ z_n = J^{-1}(\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n) \\ u_n = T_{r_n} y_n, \\ C_0 = \{z \in C : \phi(z, u_0) \leq \phi(z, x_0)\}, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ Q_0 = C, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \ n \geq 0, \end{cases}$$

where $\{\beta_n^{(1)}\}, \{\beta_n^{(2)}\}, \{\beta_n^{(3)}\}$ and $\{\alpha_n\}$ are four sequences in [0,1] satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ for all $n \geq 0$. The following conditions hold:

(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;
(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;
(iii) $\limsup_{n\to\infty} \alpha_n < 1$.

Then $\{x_n\}$ *converges strongly to* $\Pi_\Omega x_0$.

Remark 3.5 Corollary 3.4 improves and generalizes Theorem 3.1 in [15].

Next, we prove a convergence theorem for Halpern-type iterative algorithm.

Theorem 3.6 *Let E be a real uniformly smooth and 2-uniformly convex Banach space (e.g., L^p for $1 < p \leq 2$) and C be a nonempty, closed and convex subset of E. Let $W : E \rightrightarrows E^*$ be a maximal monotone operator satisfying $D(W) \subset C$ and let $J_t = (J + tW)^{-1} J$ for all $t > 0$. Let $A : C \to E^*$ be a $\gamma-$inverse strongly monotone mapping, $B : C \to E^*$ be a monotone continuous mapping. Let $\{T_n\}, \{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $\Omega := (\cap_{n=0}^\infty F(T_n)) \cap (\cap_{n=0}^\infty F(S_n)) \cap VI(C, A) \cap GMEP \cap W^{-1}0 \neq \emptyset$. Assume that $\|Ax\| \leq \|Ax - Ap\|$ for all $x \in C$ and $p \in VI(C, A)$. Suppose that $0 < a < \lambda_n < b = \frac{c^2\gamma}{2}$, where c is the constant in (2.1). Let $\{r_n\} \subset [c^*, +\infty)$ for some $c^* > 0$ and $\{t_n\} \subset (0, +\infty)$ satisfy $\liminf_{n\to\infty} t_n > 0$. Let $\{x_n\}$ be the sequence generated by the algorithm*

$$\begin{cases} x_0 \in C, \text{ chosen arbitrarily,} \\ v_n = \Pi_C J^{-1}(Jx_n - \lambda_n Ax_n), \\ z_n = J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n J_{t_n} v_n + \beta_n^{(3)} JS_n v_n), \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)Jz_n) \\ u_n = K_{r_n} y_n, \\ C_0 = \{z \in C : \phi(z, u_0) \leq \phi(z, x_0)\}, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \\ \qquad \phi(z, u_n) \leq (1 - (1-\alpha_n)\beta_n^{(1)})\phi(z, x_n) + (1-\alpha_n)\beta_n^{(1)} \phi(z, x_0)\}, \\ Q_0 = C, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \quad n \geq 0, \end{cases}$$

where J is the normalized duality mapping, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$, $\{\beta_n^{(3)}\}$ and $\{\alpha_n\}$ are sequences in [0,1] satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ for all $n \geq 0$, and, moreover, the conditions below:

(i) $\lim_{n\to\infty} \beta_n^{(1)} = 0$;
(ii) $\liminf_{n\to\infty} \beta_n^{(2)}\beta_n^{(3)} > 0$;
(iii) $\liminf_{n\to\infty}(1-\alpha_n) > 0$.

Then $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.

Proof Put $w_n = J_{t_n} v_n$. It is easy to see that C_n and Q_n are both closed and convex for each $n \geq 0$. We first show that $\Omega \subset C_n \cap Q_n$ for all $n \geq 0$.

Indeed, observe that $\Omega \subset Q_0 = C$ and from Lemmas 2.8 and 2.9, for any $p \in \Omega$, we have

$$\begin{aligned}\phi(p, u_0) &= \phi(p, K_{r_0} y_0)\\ &\leq \phi(p, y_0)\\ &= \phi(p, J^{-1}(\alpha_0 J x_0 + (1-\alpha_0) J z_0))\\ &\leq \alpha_0 \phi(p, x_0) + (1-\alpha_0)\phi(p, z_0),\\ \phi(p, z_0) &= \phi(p, J^{-1}(\beta_0^{(1)} J x_0 + \beta_0^{(2)} J T_0 J_{t_0} v_0 + \beta_0^{(3)} J S_0 v_0))\\ &\leq \beta_0^{(1)}\phi(p, x_0) + \beta_0^{(2)}\phi(p, J_{t_0} v_0) + \beta_0^{(3)}\phi(p, v_0)\\ &\leq \beta_0^{(1)}\phi(p, x_0) + \beta_0^{(2)}\phi(p, v_0) + \beta_0^{(3)}\phi(p, v_0).\end{aligned} \tag{3.30}$$

By (3.3), (3.30), we obtain

$$\phi(p, u_0) \leq \alpha_0\phi(p, x_0) + (1-\alpha_0)\phi(p, x_0) = \phi(p, x_0).$$

So $p \in C_0$. Thus $p \in C_0 \cap Q_0$. Assume that $p \in C_{n-1} \cap Q_{n-1}$ The relations (3.30) and (3.8) imply that

$$\begin{aligned}\phi(p, u_n) &\leq \phi(p, y_n)\\ &\leq \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n),\\ \phi(p, z_n) &\leq \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, T_n w_n) + \beta_n^{(3)}\phi(p, S_n v_n)\\ &\leq \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, w_n) + \beta_n^{(3)}\phi(p, v_n)\\ &\leq \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, v_n) + \beta_n^{(3)}\phi(p, v_n)\\ &\leq \beta_n^{(1)}\phi(p, x_0) + (1-\beta_n^{(1)})\phi(p, x_n).\end{aligned} \tag{3.31}$$

Hence

$$\phi(p, u_n) \leq (1-(1-\alpha_n)\beta_n^{(1)})\phi(p, x_n) + (1-\alpha_n)\beta_n^{(1)}\phi(p, x_0).$$

This implies $p \in C_n$. By the proof of Step 1 in Theorem 3.1, we have $p \in Q_n$ and so $p \in C_n \cap Q_n$. By induction, we have $\Omega \subset C_n \cap Q_n$ for all $n \geq 0$.

Similarly to the proof of Theorem 3.1, we have

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0, \tag{3.32}$$

$$\|x_{n+1} - x_n\| = 0 \tag{3.33}$$

and the sequence $\{x_n\}$ is a Cauchy sequence. Let $x^* \in C$ be the (strong) limit of $\{x_n\}$. It particularly turns out that the sequences $\{x_n\}$, $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{T_n w_n\}$ and $\{S_n v_n\}$ are all bounded. Since $x_{n+1} \in C_n$, by (3.32) and the condition (i) we have

$$\phi(x_{n+1}, u_n) \leq (1-(1-\alpha_n)\beta_n^{(1)})\phi(x_{n+1}, x_n) + (1-\alpha_n)\beta_n^{(1)}\phi(x_{n+1}, x_0) \to 0 \quad (n \to \infty).$$

Hence by Lemma 2.3, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0.$$

This and (3.33) imply that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.34}$$

Since J is uniformly continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{3.35}$$

For any $p \in \Omega$, it follows from Lemma 2.7, (3.8) and the boundedness of $\{T_n w_n\}$ and $\{S_n v_n\}$ that

$$\begin{aligned}
\phi(p, z_n) &= \|p\|^2 - 2\langle p, J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n w_n + \beta_n^{(3)} JS_n v_n)\rangle \\
&\quad + \|J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n w_n + \beta_n^{(3)} JS_n v_n)\|^2 \\
&\le \|p\|^2 - 2\beta_n^{(1)}\langle p, Jx_0\rangle - 2\beta_n^{(2)}\langle p, JT_n w_n\rangle - 2\beta_n^{(3)}\langle p, JS_n v_n\rangle \\
&\quad + \beta_n^{(1)}\|x_0\|^2 + \beta_n^{(2)}\|T_n w_n\|^2 + \beta_n^{(3)}\|S_n v_n\|^2 - \beta_n^{(2)}\beta_n^{(3)} g(\|JT_n w_n - JS_n v_n\|) \\
&= \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, T_n w_n) + \beta_n^{(3)}\phi(p, S_n v_n) - \beta_n^{(2)}\beta_n^{(3)} g(\|JT_n w_n - JS_n v_n\|) \\
&\le \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, w_n) + \beta_n^{(3)}\phi(p, v_n) - \beta_n^{(2)}\beta_n^{(3)} g(\|JT_n w_n - JS_n v_n\|) \\
&\le \beta_n^{(1)}\phi(p, x_0) + (1-\beta_n^{(1)})\phi(p, x_n) - \beta_n^{(2)}\beta_n^{(3)} g(\|JT_n w_n - JS_n v_n\|).
\end{aligned}$$

From the above inequality and (3.31), we have

$$\begin{aligned}
\phi(p, u_n) &\le \alpha_n\phi(p, x_n) + (1-\alpha_n)[\beta_n^{(1)}(\phi(p, x_0) - \phi(p, x_n)) \\
&\quad + \phi(p, x_n) - \beta_n^{(2)}\beta_n^{(3)} g(\|JT_n w_n - JS_n v_n\|)] \\
&\le \phi(p, x_n) + (1-\alpha_n)\beta_n^{(1)}(\phi(p, x_0) - \phi(p, x_n)) \\
&\quad - (1-\alpha_n)\beta_n^{(2)}\beta_n^{(3)} g(\|JT_n w_n - JS_n v_n\|).
\end{aligned}$$

Hence

$$\begin{aligned}
&(1-\alpha_n)\beta_n^{(2)}\beta_n^{(3)} g(\|JT_n w_n - JS_n v_n\|) \\
&\quad \le \phi(p, x_n) - \phi(p, u_n) + (1-\alpha_n)\beta_n^{(1)}(\phi(p, x_0) - \phi(p, x_n)) \\
&\quad \le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|p\|\|Jx_n - Ju_n\| + (1-\alpha_n)\beta_n^{(1)}(\phi(p, x_0) - \phi(p, x_n)).
\end{aligned}$$

From the conditions (i)–(iii), (3.34), (3.35) and the properties of the mapping g, we obtain

$$\lim_{n\to\infty} \|JT_n w_n - JS_n v_n\| = 0. \tag{3.36}$$

Now (3.31) together with the reasoning of (3.3) imply that

$$\begin{aligned}
\phi(p, z_n) &\le \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, v_n) + \beta_n^{(3)}\phi(p, v_n) \\
&\le \beta_n^{(1)}\phi(p, x_0) + (1-\beta_n^{(1)})[\phi(p, x_n) - 2\lambda_n\gamma\|Ax_n - Ap\|^2 + \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2] \\
&= \beta_n^{(1)}\phi(p, x_0) + (1-\beta_n^{(1)})\phi(p, x_n) + (1-\beta_n^{(1)})2\lambda_n\left(\frac{2}{c^2}\lambda_n - \gamma\right)\|Ax_n - Ap\|^2.
\end{aligned}$$

From the above inequality and (3.31), we have

$$\begin{aligned}
\phi(p, u_n) &\le \alpha_n\phi(p, x_n) + (1-\alpha_n)\phi(p, z_n) \\
&\le \alpha_n\phi(p, x_n) + (1-\alpha_n)(\beta_n^{(1)}\phi(p, x_0) + (1-\beta_n^{(1)})\phi(p, x_n) \\
&\quad + (1-\beta_n^{(1)})2\lambda_n\left(\frac{2}{c^2}\lambda_n - \gamma\right)\|Ax_n - Ap\|^2) \\
&= \phi(p, x_n) + (1-\alpha_n)\beta_n^{(1)}(\phi(p, x_0) - \phi(p, x_n)) \\
&\quad + (1-\alpha_n)(1-\beta_n^{(1)})2\lambda_n\left(\frac{2}{c^2}\lambda_n - \gamma\right)\|Ax_n - Ap\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n^{(1)})2\lambda_n\left(\gamma-\frac{2}{c^2}\lambda_n\right)\|Ax_n-Ap\|^2\\
&\leq \phi(p,x_n)-\phi(p,u_n)+(1-\alpha_n)\beta_n^{(1)}(\phi(p,x_0)-\phi(p,x_n))\\
&\leq \|x_n-u_n\|(\|x_n\|+\|u_n\|)+2\|p\|\|Jx_n-Ju_n\|\\
&\quad+(1-\alpha_n)\beta_n^{(1)}(\phi(p,x_0)-\phi(p,x_n)).
\end{aligned}$$

Hence from the conditions (i), (iii), (3.34) and (3.35), we have

$$\lim_{n\to\infty}\|Ax_n-Ap\|^2=0. \tag{3.37}$$

From Lemmas 2.1, 2.2, 2.6, (3.37) and the fact of $\|Ax\|\leq\|Ax-Ap\|$ for all $x\in C$ and $p\in \mathrm{VI}(C,A)$, we have

$$\begin{aligned}
\phi(x_n,v_n)&=\phi(x_n,\Pi_C J^{-1}(Jx_n-\lambda_n Ax_n))\\
&\leq \phi(x_n,J^{-1}(Jx_n-\lambda_n Ax_n))\\
&=V(x_n,Jx_n-\lambda_n Ax_n)\\
&\leq V(x_n,Jx_n-\lambda_n Ax_n+\lambda_n Ax_n)-2\langle J^{-1}(Jx_n-\lambda_n Ax_n)-x_n,\lambda_n Ax_n\rangle\\
&=\phi(x_n,x_n)+2\lambda_n\|J^{-1}(Jx_n-\lambda_n Ax_n)-J^{-1}Jx_n\|\|Ax_n\|\\
&\leq 2\lambda_n^2\frac{2}{c^2}\|Ax_n\|^2.\\
&\leq 2\lambda_n^2\frac{2}{c^2}\|Ax_n-Ap\|^2\to 0\quad (n\to\infty).
\end{aligned}$$

This implies that

$$\|x_n-v_n\|\to 0\quad (n\to\infty). \tag{3.38}$$

From (3.31), (3.34)–(3.35), Lemma 2.8 and the conditions (i) and (iii), for any $p\in\Omega$ we have

$$\begin{aligned}
\phi(u_n,y_n)&=\phi(K_{r_n}y_n,y_n)\\
&\leq \phi(p,y_n)-\phi(p,u_n)\\
&\leq \alpha_n\phi(p,x_n)+(1-\alpha_n)[\beta_n^{(1)}\phi(p,x_0)+(1-\beta_n^{(1)})\phi(p,x_n)]-\phi(p,u_n)\\
&=\phi(p,x_n)-\phi(p,u_n)+(1-\alpha_n)\beta_n^{(1)}(\phi(p,x_0)-\phi(p,x_n))\\
&\leq \|x_n-u_n\|(\|x_n\|+\|u_n\|)+2\|p\|\|Jx_n-Ju_n\|\\
&\quad+(1-\alpha_n)\beta_n^{(1)}(\phi(p,x_0)-\phi(p,x_n))\to 0\quad (n\to\infty).
\end{aligned}$$

So

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{3.39}$$

It follows from (3.34) and (3.39) that

$$\lim_{n\to\infty}\|x_n-y_n\|=0.$$

Since J is uniformly continuous on bounded sets,

$$\lim_{n\to\infty}\|Jx_n-Jy_n\|=0. \tag{3.40}$$

Since $y_n=J^{-1}(\alpha_n Jx_n+(1-\alpha_n)Jz_n)$, i.e., $Jy_n=\alpha_n Jx_n+(1-\alpha_n)Jz_n$. Now by virtue of (3.40) and the condition (iii), we have

$$\|Jx_n-Jz_n\|=\frac{1}{1-\alpha_n}\|Jy_n-Jx_n\|\to 0\quad (n\to\infty). \tag{3.41}$$

Since J^{-1} is uniformly continuous on bounded sets, we get

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.42}$$

By virtue of (3.38) and (3.42), we have

$$\lim_{n\to\infty} \|z_n - v_n\| = 0.$$

Since J is uniformly continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|Jz_n - Jv_n\| = 0. \tag{3.43}$$

Since $z_n = J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n w_n + \beta_n^{(3)} JS_n v_n)$, $Jz_n = \beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n w_n + \beta_n^{(3)} JS_n v_n$; it follows that

$$\begin{aligned}\|Jz_n - Jv_n\| &= \|\beta_n^{(1)}(Jx_0 - Jv_n) + \beta_n^{(2)}(JT_n w_n - Jv_n) + \beta_n^{(3)}(JS_n v_n - Jv_n)\| \\ &\geq \|\beta_n^{(2)}(JT_n w_n - Jv_n) + \beta_n^{(3)}(JS_n v_n - Jv_n)\| - \beta_n^{(1)}\|Jx_0 - Jv_n\|.\end{aligned}$$

Combining the last inequality, (3.43) and the condition (i), we obtain

$$\|\beta_n^{(2)}(JT_n w_n - Jv_n) + \beta_n^{(3)}(JS_n v_n - Jv_n)\| \leq \|Jz_n - Jv_n\| + \beta_n^{(1)}\|Jx_0 - Jv_n\| \to 0. \tag{3.44}$$

On the other hand, by using the property of norm $\|\cdot\|$, we have

$$\begin{aligned}&\|\beta_n^{(2)}(JT_n w_n - Jv_n) + \beta_n^{(3)}(JS_n v_n - Jv_n)\| \\ &= \|\beta_n^{(2)}(JT_n w_n - Jv_n) + \beta_n^{(3)}(JS_n v_n - Jv_n) + \beta_n^{(3)}(JT_n w_n - Jv_n) - \beta_n^{(3)}(JT_n w_n - Jv_n)\| \\ &= \|(\beta_n^{(2)} + \beta_n^{(3)})(JT_n w_n - Jv_n) + \beta_n^{(3)}(JS_n v_n - JT_n w_n)\| \\ &\geq \|(\beta_n^{(2)} + \beta_n^{(3)})(JT_n w_n - Jv_n)\| - \beta_n^{(3)}\|JS_n v_n - JT_n w_n\|.\end{aligned}$$

From (3.36), (3.44) and the above inequality, we obtain

$$\lim_{n\to\infty} \|(\beta_n^{(2)} + \beta_n^{(3)})(JT_n w_n - Jv_n)\| = \lim_{n\to\infty} \|(1 - \beta_n^{(1)})(JT_n w_n - Jv_n)\| = 0,$$

which together with the condition (i) imply that

$$\lim_{n\to\infty} \|JT_n w_n - Jv_n\| = 0.$$

Since J^{-1} is uniformly continuous on bounded sets, we have

$$\lim_{n\to\infty} \|T_n w_n - v_n\| = 0. \tag{3.45}$$

Similarly, we can prove that

$$\lim_{n\to\infty} \|S_n v_n - v_n\| = 0. \tag{3.46}$$

For any $p \in \Omega$, it follows from (3.8) and (3.31) that

$$\begin{aligned}\phi(p, z_n) &\leq \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, w_n) + \beta_n^{(3)}\phi(p, v_n) \\ &\leq \beta_n^{(1)}\phi(p, x_0) + \beta_n^{(2)}\phi(p, w_n) + \beta_n^{(3)}\phi(p, x_n).\end{aligned} \tag{3.47}$$

By Lemma 2.9, (3.8), (3.41)–(3.42), (3.47) and the conditions (i) and (ii), we obtain

$$\begin{aligned}
\phi(w_n, v_n) &= \phi(J_{t_n} v_n, v_n) \\
&\le \phi(p, v_n) - \phi(p, J_{t_n} v_n) \\
&\le \phi(p, x_n) - \phi(p, w_n) \\
&\le \phi(p, x_n) - \frac{1}{\beta_n^2}(\phi(p, z_n) - \beta_n^{(1)}\phi(p, x_0) - \beta_n^{(3)}\phi(p, x_n)) \\
&= \frac{1}{\beta_n^{(2)}}(\phi(p, x_n) - \phi(p, z_n)) + \frac{\beta_n^{(1)}}{\beta_n^{(2)}}(\phi(p, x_0) - \phi(p, x_n)) \\
&\le \frac{1}{\beta_n^{(2)}}(\|x_n - z_n\|(\|x_n\| + \|z_n\|) + 2\|p\|\|Jx_n - Jz_n\|) \\
&\quad + \frac{\beta_n^{(1)}}{\beta_n^{(2)}}(\phi(p, x_0) - \phi(p, x_n)) \to 0 \quad (n \to \infty),
\end{aligned}$$

which together with Lemma 2.3 imply that

$$\lim_{n\to\infty} \|w_n - v_n\| = 0. \tag{3.48}$$

From (3.45) and (3.48), we obtain

$$\lim_{n\to\infty} \|T_n w_n - w_n\| = 0. \tag{3.49}$$

Since $x_n \to x^*$ as $n \to \infty$, by (3.38) and (3.48), we have $v_n \to x^*$ and $w_n \to x^*$ as $n \to \infty$. So by (3.46) and (3.49), we conclude that $x^* \in \cap_{n=0}^{\infty} F(T_n) \cap \cap_{n=0}^{\infty} F(S_n)$.

Similarly to the proof of Step 3 in Theorem 3.1, we have $x^* \in \mathrm{VI}(C, A) \cap \mathrm{GMEP} \cap W^{-1}0$. So $x^* \in \Omega$. Similarly to the proof of Step 4 in Theorem 3.1, we get $x^* = \Pi_\Omega x_0$. □

If $A \equiv 0$, $W \equiv 0$, then similarly to the proof of Corollary 3.2, we find that the uniform smoothness and 2-uniform convexity of E can be weakened to uniform smoothness and uniform convexity of E. So from Theorem 3.6, we have the following corollary.

Corollary 3.7 *Let E be a real uniformly smooth and uniformly convex Banach space and C be a nonempty, closed and convex subset of E. Let $B : C \to E^*$ be a monotone continuous mapping. Let $\{T_n\}$, $\{S_n\}$ be two countable families of weak relatively nonexpansive mappings from C into itself such that $\Omega := (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n)) \cap GMEP \neq \emptyset$. Suppose that $0 < a < \lambda_n < b = \frac{1}{2}c^2\gamma$, where c is the constant in (2.1). Let $\{r_n\} \subset [c^*, +\infty)$ for some $c^* > 0$. Let $\{x_n\}$ be the sequence generated by the following algorithm*

$$\begin{cases}
x_0 \in C, \ \text{chosen arbitrarily}, \\
z_n = J^{-1}(\beta_n^{(1)} Jx_0 + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n), \\
y_n = J^{-1}(\alpha_n Jx_n + (1 - \alpha_n)Jz_n) \\
u_n = K_{r_n} y_n, \\
C_0 = \{z \in C : \phi(z, u_0) \le \phi(z, x_0)\}, \\
C_n = \{z \in C_{n-1} \cap Q_{n-1} : \\
\qquad \phi(z, u_n) \le (1 - (1 - \alpha_n)\beta_n^{(1)})\phi(z, x_n) + (1 - \alpha_n)\beta_n^{(1)}\phi(z, x_0)\}, \\
Q_0 = C, \\
Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\
x_{n+1} = \Pi_{C_n \cap Q_n} x_0, \ n \ge 0,
\end{cases}$$

where J is the normalized duality mapping, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$, $\{\beta_n^{(3)}\}$ and $\{\alpha_n\}$ are sequences in [0,1] satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ for all $n \ge 0$, and the following conditions:

(i) $\lim_{n\to\infty} \beta_n^{(1)} = 0$;
(ii) $\liminf_{n\to\infty} \beta_n^{(2)}\beta_n^{(3)} > 0$;
(iii) $\liminf_{n\to\infty}(1 - \alpha_n) > 0$.

Then $\{x_n\}$ *converges strongly to* $\Pi_\Omega x_0$.

Remark 3.8 Corollary 3.7 improves and extends Theorem 3.19 in [8] and Theorem 3.3 in [10].

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